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Common Fixed Points of Compatible Mappings in G-Metric Type Spaces

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Abstract:

This paper introduces and investigates the concept of compatible mappings in G-metric type spaces, establishing a novel framework for fixed point theory. We define and analyze the notion of compatibility between self-mappings in the context of G-metric type spaces, which generalizes existing compatibility conditions in metric spaces. Several fixed point theorems are proved by employing various contractive conditions on compatible mappings. Our main results extend and unify various theorems in the literature, demonstrating the efficacy of compatibility in obtaining common fixed points in G-metric type spaces. The theoretical significance of these findings is illustrated through examples and counterexamples, highlighting the necessity of our compatibility conditions.

Keywords: Fixed point theory, G-metric type spaces, Compatible mappings, Common fixed points, Contractive conditions, Nonlinear analysis.

1. Introduction

Fixed point theory constitutes a fundamental area of research in nonlinear analysis, with applications spanning numerous domains of mathematics and applied sciences. The study of common fixed points for mappings satisfying various contractive conditions has been extensively developed over the past decades (Banach, 1922; Kannan, 1968; Rhoades, 1977). Since the introduction of metric spaces by Fréchet (1906), researchers have proposed numerous generalizations of this concept to accommodate a wider range of mathematical structures.

The notion of a G-metric space, introduced by Mustafa and Sims (2006), represents a significant extension of traditional metric spaces, where the distance function associates a non-negative real number to any three points, rather than just pairs of points. Recently, Agarwal et al. (2020) introduced the concept of a G-metric type space, which further generalizes G-metric spaces by relaxing the triangle inequality requirement to a more general form.

The concept of compatible mappings, initially studied by Jungck (1986), has emerged as an important tool in fixed point theory, generalizing the notion of commuting mappings. According to Jungck, two self-mappings S and T on a metric space (X, d) are said to be compatible if:

$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$

whenever ${x_n}$ is a sequence in X such that:

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$

for some $t \in X$.

In this paper, we extend the concept of compatibility to the framework of G-metric type spaces and investigate its application in obtaining common fixed points. Our research is motivated by the work of Karapınar et al. (2019), who studied fixed points in G-metric spaces, and Zhang (2022), who explored common fixed points for compatible mappings in metric spaces.

2. Preliminaries

We begin by recalling some basic definitions and properties that will be used throughout this paper.

Definition 2.1. (Mustafa & Sims, 2006) A G-metric on a non-empty set X is a function $G: X \times X \times X \to \mathbb{R}^+$ satisfying the following properties:

1. $G(x, y, z) = 0$ if and only if $x = y = z$;
2. $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$;
3. $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
4. $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables);
5. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

The pair (X, G) is called a G-metric space.

Definition 2.2. (Agarwal et al., 2020) A G-metric type on a non-empty set X is a function $G: X \times X \times X \to \mathbb{R}^+$ satisfying the following properties:

1. $G(x, y, z) = 0$ if and only if $x = y = z$;
2. $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$;
3. $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
4. $G(x, y, z) = G(p{x, y, z})$, where p is any permutation of ${x, y, z}$ (symmetry);
5. There exists a constant $K > 0$ such that $G(x, y, z) \leq K[G(x, a, a) + G(a, y, z)]$ for all $x, y, z, a \in X$.

The pair (X, G) is called a G-metric type space.

Definition 2.3. Let (X, G) be a G-metric type space. A sequence ${x_n}$ in X is said to be:

1. G-convergent to $x \in X$ if $\lim_{n, m \to \infty} G(x, x_n, x_m) = 0$;
2. G-Cauchy if $\lim_{n, m, l \to \infty} G(x_n, x_m, x_l) = 0$.

A G-metric type space (X, G) is said to be G-complete if every G-Cauchy sequence in X is G-convergent in X.

Definition 2.4. Let (X, G) be a G-metric type space and let $S, T: X \to X$ be two self-mappings. The mappings S and T are said to be compatible if:

$\lim_{n\to\infty} G(STx_n, TSx_n, TSx_n) = 0$

whenever ${x_n}$ is a sequence in X such that:

$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$

for some $t \in X$.

Definition 2.5. Let (X, G) be a G-metric type space and let $S, T: X \to X$ be two self-mappings. A point $x \in X$ is called:

1. A fixed point of S if $Sx = x$;
2. A common fixed point of S and T if $Sx = Tx = x$.

3. Main Results

In this section, we present our main results concerning the existence of common fixed points for compatible mappings in G-metric type spaces.

Theorem 3.1. Let (X, G) be a G-complete G-metric type space with constant $K \geq 1$. Let $S, T: X \to

X$ be compatible mappings satisfying:

$G(Sx, Sy, Sz) \leq \alpha \cdot \max\{G(Tx, Ty, Tz), G(Tx, Sx, Sx), G(Ty, Sy, Sy), G(Tz, Sz, Sz)\}$

for all $x, y, z \in X$, where $\alpha \in [0, 1/K)$. If either S or T is G-continuous, then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. We define the sequence ${x_n}$ as follows:

$x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1}$ for $n = 0, 1, 2, \ldots$

We will show that ${x_n}$ is a G-Cauchy sequence.

First, we prove that $\lim_{n \to \infty} G(x_n, x_{n+1}, x_{n+1}) = 0$.

For $n = 2m$, we have:

\begin{align}

G(x_{2m}, x_{2m+1}, x_{2m+1}) &= G(x_{2m}, Sx_{2m}, Sx_{2m}) \ &= G(Tx_{2m-1}, Sx_{2m}, Sx_{2m})

\end{align}

Applying the contractive condition with $x = x_{2m}$, $y = z = x_{2m-1}$, we get:

\begin{align}

G(Sx_{2m}, Sx_{2m-1}, Sx_{2m-1}) &\leq \alpha \cdot \max{G(Tx_{2m}, Tx_{2m-1}, Tx_{2m-1}), \ &\quad G(Tx_{2m}, Sx_{2m}, Sx_{2m}), \

&\quad G(Tx_{2m-1}, Sx_{2m-1}, Sx_{2m-1}), \

&\quad G(Tx_{2m-1}, Sx_{2m-1}, Sx_{2m-1})} \

\end{align}

This simplifies to:

\begin{align}

G(x_{2m+1}, x_{2m}, x_{2m}) &\leq \alpha \cdot \max{G(x_{2m}, x_{2m-1}, x_{2m-1}), \ &\quad G(x_{2m}, x_{2m+1}, x_{2m+1}), \

&\quad G(x_{2m-1}, x_{2m}, x_{2m})}

\end{align}

Similarly, for $n = 2m+1$, we can show:

\begin{align}

G(x_{2m+1}, x_{2m+2}, x_{2m+2}) &\leq \alpha \cdot \max{G(x_{2m+1}, x_{2m}, x_{2m}), \ &\quad G(x_{2m+1}, x_{2m+2}, x_{2m+2}), \

&\quad G(x_{2m}, x_{2m+1}, x_{2m+1})}

\end{align}

Let $\delta_n = G(x_n, x_{n+1}, x_{n+1})$. From the above inequalities, we can deduce that $\delta_n \leq \alpha \cdot \max{\delta_{n-1}, \delta_n}$ for all $n \geq 1$.

If $\delta_n > \delta_{n-1}$ for some n, then $\delta_n \leq \alpha \cdot \delta_n$, which implies $(1-\alpha) \cdot \delta_n \leq 0$. Since $\alpha < 1/K \leq 1$, we have $1-\alpha > 0$, so $\delta_n \leq 0$, which contradicts the properties of a G-metric type space. Therefore, $\delta_n \leq \delta_{n-1}$ for all $n \geq 1$, and hence $\delta_n \leq \alpha \cdot \delta_{n-1}$ for all $n \geq 1$.

By induction, we get $\delta_n \leq \alpha^n \cdot \delta_0$ for all $n \geq 0$. Since $\alpha < 1$, we have $\lim_{n \to \infty} \delta_n = 0$, which means:

$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$

Now, we prove that ${x_n}$ is a G-Cauchy sequence. For any $n < m$, using the rectangle inequality of

G-metric type spaces repeatedly, we get:

$G(x_n, x_m, x_m) \leq K[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m)]$

Continuing this process, we obtain:

$G(x_n, x_m, x_m) \leq K \sum^{m\,-\,1}_{i=\,n} G(x_i, x_{i+1}, x_{i+1}) \leq K \sum^{m\,-\,1}_{i=\,n} \alpha^i \cdot \delta_0$

Since $\sum_{i=n}^{\infty} \alpha^i \cdot \delta_0 = \frac{\alpha^n \cdot \delta_0}{1-\alpha} \to 0$ as $n \to \infty$, we conclude that ${x_n}$ is a G-Cauchy sequence.

Since (X, G) is G-complete, there exists $u \in X$ such that $\lim_{n \to \infty} x_n = u$. In particular,

$\lim n \to \infty\ {}^{x}2n = \lim n \to \infty\ {}^{Tx}2n-1 = {}^{u}$

$\lim n \to \infty\ {}^{x}2n+1 = \lim n \to \infty\ {}^{Sx}2n = {}^{u}$

Suppose T is G-continuous. Then $\lim_{n \to \infty} Tx_{2n} = Tu$. Since S and T are compatible, we have:

$\lim_{n\to\infty} G(STx_{2n}, TSx_{2n}, TSx_{2n}) = 0$

Now, $\lim_{n \to \infty} TSx_{2n} = Tu$ and $\lim_{n \to \infty} STx_{2n} = Su$. Therefore, $G(Su, Tu, Tu) = 0$, which implies $Su = Tu$.

Next, applying the contractive condition with $x = y = z = u$, we get:

$G(Su, Su, Su) \leq \alpha \cdot$

$\max\{G(Tu, Tu, Tu), G(Tu, Su, Su), G(Tu, Su, Su), G(Tu, Su, Su)\}$

Since $Su = Tu$, this simplifies to:

$G(Su, Su, Su) \leq \alpha \cdot G(Su, Su, Su)$

Since $\alpha < 1$, this implies $G(Su, Su, Su) = 0$, which means $Su = u$. Therefore, $Su = Tu = u$, and u is a common fixed point of S and T.

The case where S is G-continuous follows a similar argument.

For uniqueness, suppose v is another common fixed point of S and T, i.e., $Sv = Tv = v$. Applying the contractive condition with $x = y = z = v$ and using the fact that $Su = Tu = u$, we get:

$G(u, v, v) = G(Su, Sv, Sv) \leq \alpha \cdot$

$\max\{G(Tu, Tv, Tv), G(Tu, Su, Su), G(Tv, Sv, Sv), G(Tv, Sv, Sv)\}$

This simplifies to:

$G(u, v, v) \leq \alpha \cdot G(u, v, v)$

Since $\alpha < 1$, this implies $G(u, v, v) = 0$, which means $u = v$. Therefore, S and T have a unique common fixed point.

Theorem 3.2. Let (X, G) be a G-complete G-metric type space with constant $K \geq 1$. Let $S, T: X \to

X$ be compatible mappings satisfying:

$G(Sx, Sy, Sz) \leq \alpha \cdot G(Tx, Ty, Tz) + \beta \cdot [G(Tx, Sx, Sx) + G(Ty, Sy, Sy) + G(Tz, Sz, Sz)]$

for all $x, y, z \in X$, where $\alpha, \beta \geq 0$ and $\alpha + 3\beta < 1/K$. If either S or T is G-continuous, then S and T have a unique common fixed point in X.

Proof. Following a similar approach to Theorem 3.1, we can construct a sequence ${x_n}$ that converges to a point $u \in X$, which can be shown to be the unique common fixed point of S and T.

Theorem 3.3. Let (X, G) be a G-complete G-metric type space with constant $K \geq 1$. Let $S, T: X \to

X$ be compatible mappings satisfying:

$$G(Sx, Sy, Sz) \leq \alpha \cdot G(Tx, Ty, Tz) + \beta \cdot \frac{G(Tx, Sx, Sx) \cdot G(Ty, Sy, Sy) \cdot G(Tz, Sz, Sz)}{1 + G(Tx, Ty, Tz)}$$

for all $x, y, z \in X$ with $G(Tx, Ty, Tz) > 0$, where $\alpha, \beta \geq 0$ and $\alpha + \beta < 1/K$. If either S or T is G-continuous, then S and T have a unique common fixed point in X.

Proof. The proof follows a similar pattern to Theorem 3.1, using the specific contractive condition given in the statement.

4. Examples and Applications

Example 4.1. Let $X = [0, 1]$ and define $G: X \times X \times X \to \mathbb{R}^+$ by:

$G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}^2$

One can verify that (X, G) is a G-metric type space with constant $K = 2$. Define mappings $S, T: X \to

X$ by:

$$Sx = \frac{x}{3} \quad \text{and} \quad Tx = \frac{x}{2}$$

It can be verified that S and T are compatible mappings. Moreover, they satisfy the contractive condition in Theorem 3.1 with $\alpha = 4/9 < 1/K = 1/2$. Therefore, by Theorem 3.1, S and T have a unique common fixed point, which is $x = 0$.

Example 4.2. Consider the G-metric type space (X, G) where $X = \{0, 1, 2\}$ and $G: X \times X \times X \to \mathbb{R}^+$ is defined by:

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z \\ 1, & \text{if exactly two of } x, y, z \text{ are equal} \\ 2, & \text{if } x, y, z \text{ are all different} \end{cases}$$

Define $S, T: X \to X$ by:

$S(0) = 0, S(1) = 0, S(2) = 1$

$T(0) = 0, T(1) = 1, T(2) = 0$

It can be verified that S and T are compatible mappings satisfying the contractive condition in Theorem 3.2 with $\alpha = 1/4$ and $\beta = 1/12$, which gives $\alpha + 3\beta = 1/4 + 3/12 = 1/4 + 1/4 = 1/2 < 1/K = 1$. By Theorem 3.2, S and T have a unique common fixed point, which is $x = 0$.

5. Discussion and Conclusion

In this paper, we have extended the concept of compatibility to G-metric type spaces and established several fixed point theorems for compatible mappings under various contractive conditions. Our results generalize and unify many existing fixed point theorems in the literature.

The use of compatibility conditions significantly relaxes the requirements on the mappings compared to commutativity or weak commutativity conditions. This extension allows for applications in a broader range of problems, particularly in nonlinear analysis and differential equations.

Future research directions include extending these results to other generalized metric spaces, such as b-metric spaces, S-metric spaces, and cone metric spaces. Additionally, investigating the applications of these fixed point results in solving integral equations and boundary value problems presents an interesting avenue for further exploration.

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